

PARTIAL CROSSED PRODUCT PRESENTATIONS FOR O_n AND $M_k(O_n)$ USING AMENABLE GROUPS

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ABSTRACT. The Cuntz algebra O_n is presented as a partial crossed product in which an amenable group partially acts on an abelian C^* -algebra. The partial action is related to the Cuntz groupoid for O_n and connections are made with non-self-adjoint subalgebras of O_n , particularly the Volterra nest subalgebra. These ideas are also extended to the $M_k(O_n)$ context.

1. INTRODUCTION

The purpose of this note is to present the Cuntz algebra, O_n , as a partial crossed product by an amenable group acting on an abelian C^* -algebra; to connect this presentation with the standard presentation of O_n as a groupoid C^* -algebra based on the Cuntz groupoid; and to indicate how the partial crossed product presentation ties in with certain non-self-adjoint subalgebras of O_n , most notably, the Volterra nest subalgebra. In addition, we indicate how each $M_k(O_n)$ can also be written as a partial crossed product by the partial action of an amenable group on an abelian C^* -algebra. We also connect the partial crossed product presentation with a groupoid isomorphic to the usual groupoid for $M_k(O_n)$ obtained by viewing $M_k(O_n)$ as a graph C^* -algebra.

There is nothing new about writing O_n (or $M_k(O_n)$) as a crossed product or partial crossed product. In [10] and [9], Spielberg shows that O_n (as well as other graph C^* -algebras) can be presented as a crossed product in which the group is a free product of cyclic groups. Quigg and Raeburn in [8] show that O_n is a partial crossed product using the free group, \mathbb{F}_n , on n generators. Exel, Laca, and Quigg extend this to graph C^* -algebras (with finite graph), again using \mathbb{F}_n . In all these presentations, the group used is not amenable. The group appearing in our presentation of O_n is a semi-direct product of the integers and the group \mathbb{Q}_n of n -adic rationals, and so is amenable. This immediately yields yet another proof that O_n is nuclear. The group used in the presentation of $M_k(O_n)$ is the direct product of the group $\mathbb{Q}_n \times_\delta \mathbb{Z}$ and a cyclic group, and so is also amenable. There is nothing canonical about this presentation (or about the related groupoids); but

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they do display much about the internal structure of O_n (and $M_k(O_n)$) and as such should prove useful in the study of non-self-adjoint subalgebras of these C^* -algebras.

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2. O_n AS A PARTIAL CROSSED PRODUCT

Throughout this section the positive integer $n \geq 2$ will be fixed. Let X denote the Cantor set based on $[0, 1]$ with each n -adic rational r replaced by a pair r^-, r^+ . (0 and 1 are excepted, of course, and r^- is the immediate predecessor of r^+ .) The group, G , which will partially act on the abelian C^* -algebra $C(X)$ is the semi-direct product of the additive group \mathbb{Q}_n of n -adic rationals and the integers. The action, δ , of \mathbb{Z} on \mathbb{Q}_n is given by

$$\delta_k(r) = \frac{r}{n^k}, \text{ for all } k \in \mathbb{Z}, r \in \mathbb{Q}_n.$$

Multiplication and inverses in $G = \mathbb{Q}_n \rtimes_{\delta} \mathbb{Z}$ are given by

$$\begin{aligned} (s, j)(r, k) &= (\delta_j(r) + s, j + k) = \left(\frac{r}{n^j} + s, j + k\right), \\ (r, k)^{-1} &= (-\delta_{-k}(r), -k) = (-n^k r, -k). \end{aligned}$$

Define a partial action β of G on X by

$$\beta_{(r, k)}(x) = \frac{x}{n^k} + r.$$

The domain of $\beta_{(r, k)}$ is the set of all $x \in X$ for which $\frac{x}{n^k} + r \in X$. (If x is n -adic, then $\beta_{(r, k)}(x^-) = (\frac{x}{n^k} + r)^-$ and $\beta_{(r, k)}(x^+) = (\frac{x}{n^k} + r)^+$. However, normally we drop the $+$ or $-$ superscript.)

As usual, the partial action on X induces a partial action α on $C(X)$. This is given by

$$\alpha_g(f) = f \circ \beta_g^{-1}, \quad f \in C_0(\text{dom } \beta_g), \quad g \in G.$$

While $f \circ \beta_g^{-1}$ is defined only on $\text{ran } \beta_g$, it is sometimes helpful to interpret it to be defined on all of X by declaring that it has value 0 at points not in $\text{ran } \beta_g$.

The partial crossed product $C(X) \rtimes_{\alpha} G$ is generated by polynomials in the symbols U^g with the coefficient of U^g always in $C_0(\text{ran } \beta_g) = \text{ran } \alpha_g$. We should point out that the symbols U^g will not be elements of $C(X) \rtimes_{\alpha} G$; coefficients in $C_0(\text{ran } \beta_g) = \text{ran } \alpha_g$ are mandatory. However, these ideals have units, so the elements $\chi_{\text{ran } \beta_g} U^g$ do serve as substitutes for the U^g . Be warned, however, that $\chi_{\text{ran } \beta_g} U^g \chi_{\text{ran } \beta_h} U^h$ is not generally equal to $\chi_{\text{ran } \beta_{gh}} U^{gh}$. This is a consequence of the fact that $\beta_g \circ \beta_h$ is a (usually proper) restriction

of β_{gh} . The primary algebraic formulas which are needed to verify various claims in the proof of Theorem 1 are the formulas for inversion and multiplication on monomials:

$$\begin{aligned}
 (1) \quad & (fU^g)^* = \overline{\alpha_{g^{-1}}(f)}U^{g^{-1}} \\
 & = \overline{f \circ \beta_g}U^{g^{-1}} \text{ and} \\
 (2) \quad & eU^g fU^h = \alpha_g(\alpha_{g^{-1}}(e)f)U^{gh} \\
 & = e \cdot f \circ \beta_{g^{-1}}U^{gh},
 \end{aligned}$$

We shall use primarily the variant with β in the sequel.

Remark. The group \mathbb{Q}_n embeds naturally in G . If we restrict the partial action α to \mathbb{Q}_n , then $C(X) \times_\alpha \mathbb{Q}_n$ is an n^∞ -UHF algebra. Indeed, viewed as a subalgebra of $C(X) \times_\alpha G$ this will turn out to be (after the identification of $C(X) \times_\alpha G$ as O_n) the core UHF subalgebra of O_n . This subalgebra appears in section 4 where it plays a role in identifying some non-self-adjoint subalgebras of O_n in the partial crossed product formulation.

Theorem 1. $C(X) \times_\alpha G$ is isomorphic to the Cuntz algebra O_n .

The proof of Theorem 1 will make use of the standard representation of O_n as a concrete operator algebra acting on $L^2[0, 1]$. In the discussion of this representation, all subintervals of $[0, 1]$ will have n -adic endpoints; so take the word “interval” to mean “interval with n -adic endpoints”. If ϕ is an order preserving affine transformation (hereafter abbreviated “opar”) from a subinterval D of $[0, 1]$ onto a subinterval R , then ϕ induces a partial isometry on $L^2[0, 1]$ in the usual way ($f \mapsto f \circ \phi^{-1}$) with initial and final spaces identifiable with $L^2(D)$ and $L^2(R)$. Denote this partial isometry by $T(\phi)$. In particular, for each $(r, k) \in G$ the formula for $\beta_{(r,k)}$ on the Cantor space X given above (viz. $x \mapsto \frac{x}{n^k} + r$) gives an opar from a subinterval of $[0, 1]$ to another subinterval. Using this formula, we obtain a partial action of G on $[0, 1]$, which we shall also denote by β . (The context will determine clearly which partial action is intended.) The domain of $\beta_{(r,k)}$ is, of course, the largest interval in $[0, 1]$ which is mapped again into $[0, 1]$.

For each $i = 1, \dots, n$, $\beta_{(\frac{i-1}{n}, 1)}$ is the opar from $[0, 1]$ onto $[\frac{i-1}{n}, \frac{i}{n}]$; the isometries $T_i = T(\beta_{(\frac{i-1}{n}, 1)})$ have mutually orthogonal range projections which add to I . The representation of O_n which we use is the C^* -algebra generated by T_1, \dots, T_n . For the rest of this section, O_n refers to this algebra.

Let \mathcal{F} denote the family of all opars ϕ which are restrictions of some $\beta_{(r,k)}$. If ϕ and ψ are two opars in \mathcal{F} , then $\phi \circ \psi$ denotes the opar whose domain is $\{x \in \text{dom } \psi \mid \psi(x) \in \text{dom } \phi\}$ and whose action on elements of its domain is, of course, the usual formula for composition. Observe that for all $\phi, \psi \in \mathcal{F}$, $T(\phi)T(\psi) = T(\phi \circ \psi)$. Note that there may be elements of $[0, 1]$ which are not in the domain of $\phi \circ \psi$ but which are mapped by the formula for $\phi \circ \psi$ into elements of $[0, 1]$. In particular, if $g, h \in G$ then $\beta_g \circ \beta_h$ is a restriction of β_{gh} . (Simple examples where the restriction is proper can be obtained using $h = g^{-1}$;

indeed, for some of these, $\beta_g \circ \beta_h$ is the “empty” transformation.) It is now clear that $T(\beta_g)T(\beta_h) = T(\beta_g \circ \beta_h)$ is a restriction of $T(\beta_{gh})$.

If Q is a k -fold product of isometries each of which is one of T_1, \dots, T_n , then QQ^* is a projection onto $L^2(J)$, where J is some interval of length $1/n^k$. Suitable sums of such projections yield projections associated with intervals of length p/n^k , $p \in \mathbb{Z}$, $1 \leq p \leq n^k$. If we left multiply a partial isometry $T(\beta_g)$ by such a projection, we obtain $T(\phi)$, where $\phi \in \mathcal{F}$.

Similarly, “translations” are obtained from products QR^* where each of Q and R is a word of length k in T_1, \dots, T_n . More generally, for any $\phi \in \mathcal{F}$, $T(\phi)$ can be written as a sum of words in T_1, \dots, T_n and their adjoints.

Proof of Theorem 1. For each $\phi \in \mathcal{F}$, let $S(\phi) = \chi_{\text{ran } \phi} U^g$, where g is that element of G for which ϕ is a restriction of β_g . Note: to avoid extra notation, we view ϕ as acting on $[0, 1]$ and X simultaneously. In the definition of $S(\phi)$, $\text{ran } \phi$ is a subset of X – a clopen subset of $\text{ran } \beta_g$.

For each $i = 1, \dots, n$, let

$$S_i = S\left(\beta_{\left(\frac{i-1}{n}, 1\right)}\right) = \chi_{\left[\frac{i-1}{n}, \frac{i}{n}\right]} U^{\left(\frac{i-1}{n}, 1\right)}.$$

Observe that, for each i , $\text{dom } \beta_{\left(\frac{i-1}{n}, 1\right)}$ is $[0, 1]$ and $\text{ran } \beta_{\left(\frac{i-1}{n}, 1\right)}$ is $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, so that $\chi_{\left[\frac{i-1}{n}, \frac{i}{n}\right]} \circ \beta_{\left(\frac{i-1}{n}, 1\right)} = \chi_{[0, 1]}$. Formula (1) above yields

$$S_i^* = \chi_{\left[\frac{i-1}{n}, \frac{i}{n}\right]} \circ \beta_{\left(\frac{i-1}{n}, 1\right)} U^{\left(\frac{i-1}{n}, 1\right)^{-1}} = \chi_{[0, 1]} U^{(-(i-1), -1)}.$$

We then have (using formula (2))

$$\begin{aligned} S_i^* S_i &= \chi_{[0, 1]} U^{(-(i-1), -1)} \chi_{\left[\frac{i-1}{n}, \frac{i}{n}\right]} U^{\left(\frac{i-1}{n}, 1\right)} \\ &= \chi_{[0, 1]} \cdot \chi_{\left[\frac{i-1}{n}, \frac{i}{n}\right]} \circ \beta_{\left(\frac{i-1}{n}, 1\right)} U^{(0, 0)} \\ &= \chi_{[0, 1]} U^{(0, 0)} = I. \end{aligned}$$

Thus, each S_i is an isometry. Now $\text{dom } \beta_{(-(i-1), -1)} = \left[\frac{i-1}{n}, \frac{i}{n}\right]$ and $\text{ran } \beta_{(-(i-1), -1)} = [0, 1]$, so $\chi_{[0, 1]} \circ \beta_{(-(i-1), -1)} = \chi_{\left[\frac{i-1}{n}, \frac{i}{n}\right]}$. Therefore,

$$\begin{aligned} S_i S_i^* &= \chi_{\left[\frac{i-1}{n}, \frac{i}{n}\right]} U^{\left(\frac{i-1}{n}, 1\right)} \chi_{[0, 1]} U^{(-(i-1), -1)} \\ &= \chi_{\left[\frac{i-1}{n}, \frac{i}{n}\right]} \cdot \chi_{[0, 1]} \circ \beta_{(-(i-1), -1)} U^{(0, 0)} \\ &= \chi_{\left[\frac{i-1}{n}, \frac{i}{n}\right]} U^{(0, 0)}. \end{aligned}$$

This yields $\sum_i S_i S_i^* = \chi_{[0, 1]} U^{(0, 0)} = I$. Thus, S_1, \dots, S_n are n isometries in $C(X) \times_\alpha G$ with orthogonal range projections whose sum is I . This shows that $C^*(S_1, \dots, S_n)$, the

C^* -subalgebra of $C(X) \times_\alpha G$ generated by S_1, \dots, S_n , is isomorphic to the Cuntz algebra O_n . To complete the proof we need merely show that $C^*(S_1, \dots, S_n)$ is all of $C(X) \times_\alpha G$.

With the help of formula (2) we can show that $S(\phi \circ \psi) = S(\phi)S(\psi)$, for any $\phi, \psi \in \mathcal{F}$. This implies that if $T(\phi)$ is a sum of words in the T_i and T_i^* , then $S(\phi)$ is the same sum of words in the S_i and S_i^* . From the known structure of O_n described above, we deduce that, for any interval J , $\chi_J U^{(0,0)} = \chi_J I$ is in the (non-closed) $*$ -algebra generated by S_1, \dots, S_n and that, for each $g \in G$, $\chi_{\text{ran } \beta_g} U^g$ is also in this algebra. The first of these two facts implies that $f U^{(0,0)} = f I \in C^*(S_1, \dots, S_n)$, for all $f \in C(X)$. This, combined with the second fact, shows that $f U^g \in C^*(S_1, \dots, S_n)$ for all $f \in C_0(\text{ran } \beta_g)$. It now follows immediately that $C(X) \times_\alpha G \subseteq C^*(S_1, \dots, S_n)$ and the proof is complete. \square

3. THE CONNECTION BETWEEN THE PARTIAL ACTION AND THE GROUPOID

As we have seen, the group G and the partial action studied in section 2 are suggested by the Cuntz groupoid model for O_n and the standard representation of O_n acting on $L^2[0, 1]$. It is not surprising that in this situation the groupoid and the partial action determine each other. General theorems about this connection can be found in [1, 2]. The theorem in [2] does not apply directly to the partial action of G on X and the Cuntz groupoid, since the partial action is not freely acting and the Cuntz groupoid is not principal. In contrast to the two references cited, the content of this section is naive: we simply show directly how the partial action of G determines the Cuntz groupoid and vice versa. This connection does follow Theorem 5.1 in [2] in spirit.

When working with the Cuntz groupoid it is more convenient to regard X as the space of all sequences with entries from $\{0, \dots, n-1\}$. The correspondence $(a_1, a_2, \dots) \leftrightarrow \sum \frac{a_i}{n^i}$ connects the two different representations of the Cantor space X . (Sequences with a tail of 0's or with a tail of $n-1$'s correspond to n -adic rationals with a superscript $-$ or $+$, of course.) As a set, the Cuntz groupoid \mathcal{G} is the set of all triples (x, k, y) where $x, y \in X$ and $x_{i+k} = y_i$ for all sufficiently large i .

3.1. From partial action to groupoid. This is the easy direction. For each $(r, k) \in G$, define

$$\text{"graph"}(\beta_{(r,k)}) = \{(x, k, y) \mid y \in \text{dom } \beta_{(r,k)} \text{ and } x = \beta_{(r,k)}(y)\}.$$

and let

$$\mathcal{G} = \bigcup_{(r,k) \in G} \text{"graph"}(\beta_{(r,k)}).$$

The topology on \mathcal{G} is generated by all sets of the form $\{(x, k, y) \mid x \in U \text{ and } y \in V\}$, where V is an open subset of $\text{dom } \beta_{(r,k)}$ and $U = \beta_{(r,k)}(V)$. It is straightforward to check that \mathcal{G} is the Cuntz groupoid.

3.2. From groupoid to partial action. Here we need to define a G -valued cocycle c so that the “graph”($\beta_{(r,k)}$) turns out to be $c^{-1}(r,k)$. An element $(x,k,y) \in \mathcal{G}$ clearly determines k ; we need to determine r also so that $(x,k,y) \in \text{“graph”}\beta_{(r,k)}$. For this purpose, we use the standard function which converts k -tuples from $\{0,1,\dots,n-1\}$ into n -adic rationals in $[0,1]$. For $\lambda = (\lambda_1, \dots, \lambda_{|\lambda|})$, define

$$s(\lambda) = \frac{\lambda_1}{n} + \dots + \frac{\lambda_{|\lambda|}}{n^{|\lambda|}}.$$

For two finite sequences λ and μ , define

$$r(\lambda, \mu) = s(\lambda) - \frac{s(\mu)}{n^k}.$$

Finally define a cocycle $c: \mathcal{G} \rightarrow G$ as follows: with $x = \lambda z$ and $y = \mu z$ as above, and $(x,k,y) \in \mathcal{G}$, define

$$c(x,k,y) = (r(\lambda, \mu), k).$$

Note that on the set $Z(\lambda, \mu) = \{(\lambda z, |\lambda| - |\mu|, \mu z) \mid z \in X\}$ c has the constant value $(r(\lambda, \mu), |\lambda| - |\mu|)$. Since these sets form a basis for the topology on \mathcal{G} , c is locally constant.

We do need to show that c is a cocycle; for that purpose the following concatenation formula is useful. For finite strings η, ζ ,

$$s(\eta\zeta) = s(\eta) + \frac{s(\zeta)}{n^{|\eta|}}.$$

Let $(x,k,y) \in Z(\lambda, \mu)$ and $(y,l,z) \in Z(\gamma, \delta)$. Assume that this is a composable pair, which requires that one of μ and γ extend the other. Say, for the sake of argument, that $\gamma = \mu\gamma'$. (The other case can be handled in a similar way.) We have

$$\begin{aligned} c(x,k,y) &= \left(s(\lambda) - \frac{s(\mu)}{n^k}, k \right) \text{ and} \\ c(y,l,z) &= \left(s(\gamma) - \frac{s(\delta)}{n^l}, l \right). \end{aligned}$$

The product of these two elements of G is

$$\left(s(\lambda) - \frac{s(\mu)}{n^k} + \frac{s(\gamma)}{n^k} - \frac{s(\delta)}{n^{k+l}}, k+l \right).$$

Now

$$\begin{aligned} c(x,k+l,z) &= \left(s(\lambda\gamma') - \frac{s(\delta)}{n^{k+l}}, k+l \right) \\ &= \left(s(\lambda) + \frac{s(\gamma')}{n^{|\lambda|}} - \frac{s(\delta)}{n^{k+l}}, k+l \right). \end{aligned}$$

Therefore, we need to show that

$$\frac{1}{n^k}(s(\gamma) - s(\mu)) = \frac{1}{n^{|\lambda|}}s(\gamma').$$

Now, by the concatenation property,

$$s(\gamma) = s(\mu) + \frac{1}{n^{|\mu|}}s(\gamma');$$

so

$$s(\gamma) - s(\mu) = \frac{1}{n^{|\mu|}}s(\gamma')$$

and

$$\frac{1}{n^k}(s(\gamma) - s(\mu)) = \frac{1}{n^{k+|\mu|}}s(\gamma') = \frac{1}{n^{|\lambda|}}s(\gamma').$$

This establishes the cocycle property for c .

Finally, we show that for each $(r, k) \in G$, $c^{-1}(r, k)$ is the “graph” of a partial homeomorphism on X . Let

$$\begin{aligned} x &= \lambda z, \\ y &= \mu z, \\ k &= |\lambda| - |\mu|, \\ r &= r(\lambda, \mu). \end{aligned}$$

We then have

$$\begin{aligned} x &\longleftrightarrow s(\lambda) + \frac{1}{n^{|\lambda|}} \sum \frac{z_i}{n^i}, \\ y &\longleftrightarrow s(\mu) + \frac{1}{n^{|\mu|}} \sum \frac{z_i}{n^i} \end{aligned}$$

and

$$r = s(\lambda) - \frac{s(\mu)}{n^k}.$$

Calculate:

$$\begin{aligned} \beta_{(r,k)}(y) &= r + \frac{y}{n^k} \\ &= s(\lambda) - \frac{s(\mu)}{n^k} + \frac{1}{n^k} \left(s(\mu) + \frac{1}{n^{|\mu|}} \sum \frac{z_i}{n^i} \right) \\ &= s(\lambda) + \frac{1}{n^{k+|\mu|}} \sum \frac{z_i}{n^i} \\ &= s(\lambda) + \frac{1}{n^{|\lambda|}} \sum \frac{z_i}{n^i} \\ &= x. \end{aligned}$$

This ties together the cocycle and the partial action.

Remark. Different pairs λ and μ may yield the same value for $r = r(\lambda, \mu)$. But the conclusion $\beta_{(r,k)}(y) = x$ is independent of λ and μ ; it is valid for any (x, k, y) with $c(x, k, y) = (r, k)$.

4. SUBALGEBRAS

One reason for focusing on the presentation of O_n as a partial action by $\mathbb{Q}_n \times_\delta \mathbb{Z}$ on X is the possibility that this may provide further insight into (non-self-adjoint) subalgebras of O_n . Some of these algebras have been studied in [4] and, in the more general context of graph C^* -algebras, in [5]. In this section we describe a couple of these subalgebras in terms of the partial crossed product presentation.

The simplest way to obtain subalgebras is to generate them by monomials associated with a subset of G . If P is a subset of G , let $B(P)$ be the closure in $C(X) \times_\alpha G$ of the set of polynomials of the form $\sum_{g \in P} f_g U^g$. If P is closed under multiplication, then $B(P)$ is a subalgebra of O_n ; if P is closed under inversion then $B(P)$ is self-adjoint.

So, for example, if $P = \{(r, 0) \mid r \in \mathbb{Q}_n\}$, then $B(P)$ is the canonical n^∞ -UHF subalgebra of O_n . If $P = \{(r, 0) \mid r \geq 0\}$, then $B(P)$ is the refinement TAF subalgebra of the canonical UHF subalgebra. These subalgebras are not very interesting as subalgebras of O_n . For a more interesting example, let $P = \{(r, k) \mid k > 0 \text{ or } k = 0 \text{ and } r \geq 0\}$. Then $B(P)$ is a strongly maximal triangular subalgebra of O_n whose diagonal is the canonical masa in the canonical UHF subalgebra of O_n . Clearly, $B(P)$ is a superalgebra of the refinement TAF algebra just mentioned. Also, it is shown in [4] that $B(P)$ is semisimple.

The Volterra nest subalgebra \mathcal{V} of O_n is another interesting example of a non-self-adjoint subalgebra of O_n . This was first introduced by Power in [7], studied using groupoid techniques in [4] and extended to the graph C^* -algebra context in [5]. In the standard representation of O_n acting on $L^2[0, 1]$, \mathcal{V} is just the intersection of O_n and the usual Volterra nest subalgebra acting on $L^2[0, 1]$. We now proceed to identify \mathcal{V} in partial crossed product language. The reader is referred to [4] for the groupoid theoretic description of the spectrum of \mathcal{V} .

For each n -adic r (greater than 0) in X , let p_r denote the characteristic function of $[0, r^-]$. The elements $p_r U^0$ form a nest of projections in $C(X) \times_\alpha G = O_n$. This nest (with 0 adjoined) is the Volterra nest in O_n .

Let g be an element in G and let I be a clopen interval contained in $\text{dom } \beta_g$. Since $\chi_I \circ \beta_{g^{-1}} = \chi_{\beta_g(I)}$, we have

$$\chi_{\text{ran } \beta_g} U^g \chi_I U^0 = \chi_{\beta_g(I)} U^g.$$

Suppose, further, that $\beta_g(t) \leq t$, for all $t \in I$. Then $\chi_{\beta_g(I)} U^g$ leaves invariant all the Volterra projections $p_r U^0$. Indeed,

$$(p_r U^0)^\perp \chi_{\beta_g(I)} U^g p_r U^0 = p_r^\perp \chi_{\beta_g(I)} \cdot (p_r \circ \beta_{g^{-1}}) U^g,$$

so we just have to show that

$$p_r^\perp(t) \chi_{\beta_g(I)}(t) p_r(\beta_{g^{-1}}(t)) = 0, \text{ for all } t.$$

If $t \notin \beta_g(I)$, this expression is certainly 0. If $\beta_{g^{-1}}(t) > r$, it is also zero. So assume $t \in \beta_g(I)$ and $\beta_{g^{-1}}(t) \leq r$. Then $\beta_{g^{-1}}(t) \in I$ and $t = \beta_g(\beta_{g^{-1}}(t)) \leq \beta_{g^{-1}}(t) \leq r$. But then $p_r^\perp(t) = 0$. Thus the expression is zero for all values of t .

For each $(r, k) \in G$, let $I_{(r,k)}$ be the maximal clopen interval contained in $\text{dom } \beta_{(r,k)}$ with the property that $\beta_{(r,k)}(t) \leq t$ for all $t \in I_{(r,k)}$. It is possible that $I_{(r,k)}$ is empty when $\text{dom } \beta_{(r,k)}$ isn't; it is also possible that $I_{(r,k)} = \text{dom } \beta_{(r,k)}$. When $I_{(r,k)}$ is a proper subinterval of $\text{dom } \beta_{(r,k)}$, there is an n -adic number x such that $\beta_{(r,k)}(x) = x$ (more accurately, $\beta_{(r,k)}(x^-) = x^-$ and $\beta_{(r,k)}(x^+) = x^+$). In this case, either x^- will be the right hand endpoint of $I_{(r,k)}$ or x^+ will be the left hand endpoint.

Remark. It is not hard to work out when $I_{(r,k)} = \text{dom } \beta_{(r,k)}$. Since $\text{dom } \beta_{(r,k)} = \emptyset$ when $r \geq 1$ we assume $r < 1$. Then $I_{(r,k)} = \text{dom } \beta_{(r,k)}$ if, and only if, one of the following two conditions holds:

- (1) $k \geq 0$ and $r \leq 0$,
- (2) $k < 0$ and $r \leq 1 - \frac{1}{n^k}$.

For each $(r, k) \in G$, define $T_{(r,k)} = \chi_{\beta_{(r,k)}(I_{(r,k)})} U^{(r,k)}$. By the comments above, $T_{(r,k)} \in \mathcal{V}$ for all $(r, k) \in G$. In fact, $\{T_{(r,k)} \mid (r, k) \in G\}$ generates \mathcal{V} . This can be proven by showing that

$$\sigma(\mathcal{V}) = \bigcup_{(r,k) \in G} \sigma(T_{(r,k)}),$$

where σ denotes the spectrum in the groupoid. The description of $\sigma(\mathcal{V})$, which is given in [4], is a bit involved and the verification of the equality above is straightforward, so we omit the details.

5. $M_k(O_n)$ AS A PARTIAL CROSSED PRODUCT

$M_k(O_n)$ can also be written as a partial crossed product by a partial action of an amenable group on an abelian algebra. As mentioned in the introduction, algebras more general than $M_k(O_n)$ are known to be partial crossed products (with non-amenable groups). But the construction described in this section may prove useful in the study of subalgebras of $M_k(O_n)$.

Let X be the n -adic Cantor space and $G = \mathbb{Q}_n \times_\delta \mathbb{Z}$, as in section 2. Let $S_k = \{0, 1, \dots, k-1\}$. The spectrum of the abelian algebra used in the partial crossed product construction will be the Cartesian product $Y = X \times S_k$ and the group will be the product group $H = G \times \mathbb{Z}$. The partial action is given by

$$\beta_{(r,j,p)}(x, t) = \left(\frac{x}{n^j} + r, t + p \right).$$

The domain of $\beta_{(r,j,p)}$ is $\{(x, t) \in Y \mid \frac{x}{n^j} + r \in X \text{ and } t + p \in S_k\}$. Let α be the partial action on $C(Y)$ induced by β in the usual way.

The partial action β is obviously built from the partial action of G on X in section 2, which we shall now denote by β^1 and a partial action β^2 of \mathbb{Z} on S_k given by $\beta_p^2(t) = t + p$ on the obvious domain. It is well known that $C(S_k) \times_{\alpha^2} \mathbb{Z} \cong M_n$, where α^2 is β^2 transferred to $C(S_k) \cong D_n$.

The fact that $C(Y) \times_{\alpha} H \cong M_k(O_n)$ follows immediately from Theorem 2 below.

In the special case when each C_i is abelian, and so of the form $C(X_i)$, and the action of each G_i is topologically free, a shorter proof based on Theorem 2.6 of [3] is available. These conditions are satisfied by the algebras and partial actions used for $M_k(O_n)$ and the shorter proof is sketched in a remark after the proof of Theorem 2.

Theorem 2. *For $i = 1, 2$, let C_i be a nuclear C^* -algebra, G_i an amenable group, and α^i a partial action of G_i on C_i . Let $\alpha = \alpha^1 \otimes \alpha^2$ be the partial action of $G_1 \times G_2$ on $C_1 \otimes C_2$ defined on elementary tensors by*

$$\alpha_{(g_1, g_2)}(c_1 \otimes c_2) = \alpha_{g_1}^1 \otimes \alpha_{g_2}^2(c_1 \otimes c_2) = \alpha_{g_1}^1(c_1) \otimes \alpha_{g_2}^2(c_2), \quad c_1 \in \text{dom } \alpha_{g_1}^1, \quad c_2 \in \text{dom } \alpha_{g_2}^2.$$

Let $A_1 = C_1 \times_{\alpha^1} G_1$, $A_2 = C_2 \times_{\alpha^2} G_2$ and $A_3 = (A_1 \otimes A_2) \times_{\alpha} (G_1 \times G_2)$. Then $A_1 \otimes A_2 \cong A_3$.

Proof. All the groups are amenable; by [6, Prop. 4.2] the partial crossed products are isomorphic to the reduced partial crossed products. In [6] McClanahan shows how to construct from a faithful representation of a C^* -algebra a faithful representation of the reduced partial crossed product. We use this construction to prove the theorem. First, we summarize (with the suppression of details) McClanahan's construction.

Let K be a group with a partial action α on a C^* -algebra C . Let $\pi: C \rightarrow \mathcal{B}(\mathcal{H})$ be a representation. For each $g \in K$, there is a representation $\pi_g: \text{ran } \alpha_g \rightarrow \mathcal{B}(\mathcal{H})$ defined by $\pi_g(a) = \pi(\alpha_{g^{-1}}(a))$. This is extended to a representation of all of A (in a unique way) by means of an approximate unit u_λ in $\text{ran } \alpha_g$: $\pi'_g(a) = \lim \pi_g(u_\lambda a)$. (The limit is taken in the strong operator topology.) A "regular" representation $\tilde{\pi}$ of A acting on $\ell^2(K, \mathcal{H})$ is defined by $(\tilde{\pi}(a)\xi)(g) = \pi'_g(a)\xi(g)$. Let λ denote the left regular representation of K acting on $\ell^2(K, \mathcal{H})$. Finally, define $\tilde{\pi} \times \lambda$ by specifying the action on monomials: $\tilde{\pi} \times \lambda(f_g U^g) = \tilde{\pi}(f_g) \lambda_g$.

McClanahan shows that when π is faithful, then $\tilde{\pi} \times \lambda$ is a faithful representation of the reduced partial crossed product. If the group is amenable, then $\tilde{\pi} \times \lambda$ is a faithful representation of the partial crossed product.

We shall now apply this to faithful representations π_1 and π_2 of C_1 and C_2 acting on \mathcal{H}_1 and \mathcal{H}_2 . Let $\pi = \pi_1 \otimes \pi_2$. This is a faithful representation of $C_1 \otimes C_2$ acting on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \cong \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, given on monomials by $\pi(c_1 \otimes c_2) = \pi_1(c_1) \otimes \pi_2(c_2)$.

We claim that $\tilde{\pi} = \tilde{\pi}_1 \otimes \tilde{\pi}_2$. (Both of these representations act on the Hilbert space $\ell^2(G_1 \times G_2, \mathcal{H}_1 \otimes \mathcal{H}_2)$). To verify this, begin with $(g_1, g_2) \in G_1 \times G_2$. Note that $\text{ran } \alpha_{(g_1, g_2)} =$

$\text{ran } \alpha_{g_1}^1 \otimes \text{ran } \alpha_{g_2}^2$. Apply $\pi_{(g_1, g_2)}$ to an elementary tensor $c_1 \otimes c_2$ in its domain:

$$\begin{aligned} \pi_{(g_1, g_2)}(c_1 \otimes c_2) &= \pi(\alpha_{(g_1, g_2)}^{-1}(c_1 \otimes c_2)) \\ &= \pi(\alpha_{g_1}^{-1}(c_1) \otimes \alpha_{g_2}^{-1}(c_2)) \\ &= \pi_1(\alpha_{g_1}^{-1}(c_1)) \otimes \pi_2(\alpha_{g_2}^{-1}(c_2)) \\ &= (\pi_1)_{g_1}(c_1) \otimes (\pi_2)_{g_2}(c_2) \\ &= ((\pi_1)_{g_1} \otimes (\pi_2)_{g_2})(c_1 \otimes c_2). \end{aligned}$$

It follows that $\pi_{(g_1, g_2)} = (\pi_1)_{g_1} \otimes (\pi_2)_{g_2}$ on $\text{ran } \alpha_{(g_1, g_2)}$.

Take an approximate unit for $\text{ran } \alpha_{(g_1, g_2)}$ consisting of the tensor product of approximate units for $\text{ran } \alpha_{g_1}^1$ and $\text{ran } \alpha_{g_2}^2$ and strong limits to get

$$\pi'_{(g_1, g_2)} = (\pi'_1)_{g_1} \otimes (\pi'_2)_{g_2}.$$

Now, for any elementary tensors $\xi_1 \otimes \xi_2$ in $\ell^2(G_1, \mathcal{H}_1) \otimes \ell^2(G_2, \mathcal{H}_2)$ and $c_1 \otimes c_2$ in $C_1 \otimes C_2$, we have

$$\begin{aligned} (\tilde{\pi}(c_1 \otimes c_2)\xi_1 \otimes \xi_2)(g_1, g_2) &= \pi'_{(g_1, g_2)}(c_1 \otimes c_2)(\xi_1 \otimes \xi_2)(g_1, g_2) \\ &= (\pi'_1)_{g_1}(c_1)\xi_1(g_1) \otimes (\pi'_2)_{g_2}(c_2)\xi_2(g_2) \\ &= (\tilde{\pi}_1(c_1)\xi_1)_{g_1} \otimes (\tilde{\pi}_2(c_2)\xi_2)_{g_2} \\ &= ((\tilde{\pi}_1 \otimes \tilde{\pi}_2)(c_1 \otimes c_2)(\xi_1 \otimes \xi_2))(g_1, g_2). \end{aligned}$$

It follows that $\tilde{\pi}$ and $\tilde{\pi}_1 \otimes \tilde{\pi}_2$ agree everywhere.

Let λ_1 and λ_2 be the left regular representations of G_1 and G_2 on $\ell^2(G_1, \mathcal{H}_1)$ and $\ell^2(G_2, \mathcal{H}_2)$. Then $\lambda = \lambda_1 \otimes \lambda_2$ is the left regular representation of $G_1 \times G_2$ acting on $\ell^2(G_1 \times G_2, \mathcal{H}_1 \otimes \mathcal{H}_2)$. ($\lambda(g_1, g_2) = \lambda_1(g_1) \otimes \lambda_2(g_2)$.) We then have

$$\tilde{\pi} \times \lambda = (\tilde{\pi}_1 \times \lambda_1) \otimes (\tilde{\pi}_2 \times \lambda_2).$$

Since $\tilde{\pi}_i \times \lambda_i$ is a faithful representation of $C(X_i) \times_{\alpha_i} G_i$ for $i = 1, 2$ and $\tilde{\pi} \times \lambda$ is a faithful representation of $C(X_1 \times X_2) \times_{\alpha^1 \otimes \alpha^2} (G_1 \times G_2)$, the theorem follows. \square

Remark. When each C_i is abelian and each partial action is topologically free, Theorem 2.6 of [3] permits a simpler proof of Theorem 2. (“Topologically free” is defined in terms of the dual partial action on the spectrum of the C*-algebra: if β is such a dual action on X , β is *topologically free* if, for each g not the group identity, the set of fixed points of β_g has empty interior).

Here is a sketch of the simpler proof. For each i , let (π_i, v_i) be a covariant representation of (C_i, G_i, α_i) such that the corresponding representation λ_i of A_i is faithful. This implies that each π_i is faithful. Since the partial crossed product algebras are nuclear, $\lambda_1 \otimes \lambda_2$ is a faithful representation of $A_1 \otimes A_2$. It is straightforward, albeit a little tedious, to check that $(\pi_1 \otimes \pi_2, v_1 \otimes v_2)$ is a covariant representation of $(C_1 \otimes C_2, G_1 \times G_2, \alpha_1 \otimes \alpha_2)$. Let λ_3 be the corresponding representation of A_3 . Furthermore, if $a_i U^{g_i}$ are monomials in A_i ,

then another routine calculation shows that $(\lambda_1 \otimes \lambda_2)(a_1 U^{g_1} \otimes a_2 U^{g_2}) = \lambda_3(a_1 \otimes a_2 U^{(g_1, g_2)})$. It follows that $\lambda_1 \otimes \lambda_2(A_1 \otimes A_2) = \lambda_3(B_3)$. Finally, $\pi_1 \otimes \pi_2$ is a faithful representation of $C_1 \otimes C_2$, since each of π_1 and π_2 are faithful. By Theorem 2.6 in [3], λ_3 is faithful. It follows immediately that $A_1 \otimes A_2 \cong A_3$. The proof in Theorem 2 above is the same argument applied to the regular representations.

It is also possible to prove that $C(Y) \times_\alpha H \cong M_k(O_n)$ directly. We shall sketch this briefly, omitting most of the details.

Let S_1, \dots, S_n be a set of generating isometries for O_n . Let E_{ij} be matrix units for M_k . Then the following elements of $M_k(O_n) \cong O_n \otimes M_k$ generate $M_k(O_n)$;

$$\begin{aligned} T_i &= S_i \otimes E_{11}, \quad i = 1, \dots, n-1, \\ T_n &= S_n \otimes E_{1n}, \\ R_i &= I \otimes E_{i+1,i}, \quad i = 1, \dots, k-1. \end{aligned}$$

$M_k(O_n)$ is, of course, a graph C^* -algebra. One simple graph which yields $M_k(O_n)$ has k vertices and n loops. At one vertex, say v_1 , there are $n-1$ edges which are self-loops (i.e., v_1 is both the source and the range for each of these edges.) In addition, there is one more cycle made up of k edges which run successively from v_1 to v_2 to v_3 , etc. and finally from v_k back to v_1 . The generators $\{T_i, R_j\}$ satisfy the Cuntz-Krieger relations for this graph. (T_1, \dots, T_{n-1} correspond to the $n-1$ self-loops; R_1, \dots, R_{k-1}, T_n correspond to the cycle of length k).

Any set of partial isometries which satisfies this set of Cuntz-Krieger relations will generate an algebra isomorphic to $M_k(O_n)$, since this algebra satisfies the Cuntz-Krieger uniqueness theorem. So, to identify $C(Y) \times_\alpha H$ as $M_k(O_n)$, we need merely find in $C(Y) \times_\alpha H$ a set of partial isometries which satisfy these Cuntz-Krieger relations and which also generate $C(Y) \times_\alpha H$ as a C^* -algebra.

Here are a set of such generators:

$$\begin{aligned} T_i &= \chi_{[\frac{i-1}{n}, \frac{i}{n}] \times \{0\}} U^{(\frac{i-1}{n}, 1, 0)}, \quad i = 1, \dots, n-1, \\ T_n &= \chi_{[\frac{n-1}{n}, 1] \times \{0\}} U^{(\frac{n-1}{n}, 1, 1-k)}, \\ R_i &= \chi_{[0, 1] \times \{i\}} U^{(0, 0, 1)}, \quad i = 1, \dots, k-1. \end{aligned}$$

The projections which correspond to the vertices in the graph described above are

$$P_{v_i} = \chi_{[0, 1] \times \{i-1\}} U^{(0, 0, 0)}, \quad i = 1, \dots, k.$$

Routine calculations show that:

- (1) For each $i = 1, \dots, n-1$, the initial space of T_i is P_{v_1} and the final space is $\chi_{[\frac{i-1}{n}, \frac{i}{n}] \times \{0\}} U^{(0, 0, 0)}$.
- (2) For each $i = 1, \dots, k-1$, the initial space of R_i is P_{v_i} and the final space is $P_{v_{i+1}}$.

(3) The initial space of T_n is P_{v_n} and the final space is $\chi_{[\frac{n-1}{n}, \frac{i}{n}] \times \{0\}} U^{(0,0,0)}$.

The $M_k(O_n)$ –Cuntz-Krieger relations for these partial isometries follow immediately.

The calculations that $\{T_i, R_j\}$ generate $C(Y) \times_\alpha H$ as a C^* -algebra are a pain; they are best avoided entirely.

6. THE GROUPOID–PARTIAL ACTION CONNECTION FOR $M_k(O_n)$.

As in section 3, we can construct a groupoid model for $M_k(O_n)$ from the partial action of H on Y , and vice versa. Since $M_k(O_n)$ is a graph C^* -algebra, the usual groupoid model is the one based on path space with shift equivalence. For the purposes of this section, however, it is better to use a slightly different (but isomorphic) model.

With X the n -adic Cantor space, we take

$$\mathcal{G} = \{((\mu z, j), p, (\nu z, h)) \mid \mu z, \nu z \in X, j, h \in S_k, |\mu| - |\nu| = p\}.$$

The groupoid operations and topology are as expected.

Begin with the partial crossed product. Let $(r, j, p) \in H = (\mathbb{Q}_n \times_\delta \mathbb{Z}) \times \mathbb{Z}$. If $x \in X$ and $i \in S_k$ satisfy $\frac{x}{n^j} + r \in X$ and $i + p \in S_k$ then $(x, i) \in \text{dom } \beta_{(r,j,p)}$. Let $(y, q) = (\frac{x}{n^j} + r, i + p) = \beta_{(r,j,p)}(x, i)$. Define “graph” $\beta_{(r,j,p)}$ to be all such triples $((x, i), p, (y, q))$ and we obtain

$$\mathcal{G} = \bigcup_{(r,j,p) \in H} \text{“graph” } \beta_{(r,j,p)}.$$

This indicates the passage from the partial action to the groupoid.

For the other direction we need an H valued cocycle defined on \mathcal{G} . Simply define

$$c((x, i), j, (y, q)) = (r, j, p),$$

where $r = y - x = \frac{x}{n^j} - x$ and $p = q - i$. This cocycle is locally constant; furthermore, given $(r, j, p) \in H$, $c^{-1}(r, j, p)$ determines β : if $((x, i), j, (y, q)) \in c^{-1}(r, j, p)$ define $(x, i) \xrightarrow{\beta_{(r,j,p)}} (y, q)$ to get the partial action.

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